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# RESTRICTING A SCHAUDER BASIS TO A SET OF POSITIVE MEASURE

BY

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ABSTRACT. Let $\{f_n\}$ be an orthonormal system of functions on $[0, 1]$ containing a subsystem $\{f_{n_k}\}$ for which (a) $f_{n_k} \to 0$ weakly in $L_2$, and (b) given $E \subset [0, 1]$, $|E| > 0$, $\operatorname{Lim\,Inf} \int_E |f_{n_k}(x)|\,dx > 0$. There then exists a subsystem $\{g_n\}$ of $\{f_n\}$ such that for any set $E$ as above, the linear span of $\{g_n\}$ in $L_1(E)$ is not dense.

For every set $E$ as above, there is an element of $L_p(E)$, $1 < p < \infty$, whose Walsh series expansion converges conditionally and an element of $L_1(E)$ whose Haar series expansion converges conditionally.

1. Gaposhkin, in a discussion of certain properties of "lacunary" systems of functions, see [G], makes extensive use of the following notion: A sequence $\{f_n\}$ of real-valued functions on $[0, 1]$ is a "Riesz system" if the following estimates hold:

$$A_1\left(\sum_1^N C_n^2\right)^{1/2} \le \int_0^1\left|\sum_1^N C_n f_n(x)\right| dx \le A_2\left(\sum_1^N C_n^2\right)^{1/2}, \quad N \ge 1,$$

where the constants $0 < A_1 \le A_2$ are independent of the choice of $\{C_n\}$ and of $N$. An inequality of this type is used as a definition of the term "lacunary" in [KP] in an examination of complemented subspaces of the $L_p$ spaces. Moreover, the classical definition of a lacunary trigonometric system is used primarily to insure that such an inequality obtains [Zy, p. 203]. One difficulty in the application of this notion is finding enough Riesz systems in context. To this end, a sequence $\{f_n\}$ is said to have "property (B)" if there is a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to 0$ weakly in $L_2$ and $\operatorname{Lim\,Inf} \int_0^1 |f_{n_k}(x)|\,dx > 0$.

A sequence $\{f_n\}$ is said to have "property (B′)" if there is a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to 0$ weakly in $L_2$ and $\operatorname{Lim\,Inf} \int_E |f_{n_k}(x)|\,dx > 0$ whenever $E \subset (0, 1)$ and $|E| > 0$. The Lemma 1.2.6 of [G] shows that any sequence of functions having property (B) contains a subsequence that is a Riesz system. Lemma 1.2.6′ of [G] shows that for any sequence of functions $\{f_n\}$ having property (B′) there is a subsequence $\{f_{n_k}\}$ for which the following obtains: given $E \subset [0, 1]$ and $|E| > 0$, there is a $k_0 = k_0(E)$ such that $\{f_{n_k}\}_{k > k_0}$ is a Riesz system when the $f_{n_k}$ are restricted to the set $E$.

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In [PZ] Price and Zink, by taking advantage of the symmetry properties of the Rademacher functions, conclude that for any set $E \subset (0, 1)$, $|E| > 0$, the closed linear span of the Rademacher functions in $L_2(E)$ is not all of $L_2(E)$. We offer another explanation for this phenomenon by noting that the conclusion of Lemma 1.2.6′ (cited above) applies to the Rademacher system. We also obtain a generalized form of the Khintchine inequality [K, p. 130] that is of use in the next section.

Let $\|f\| = \int_0^1 |f(t)|\,dt$ and $\|f\|_E = \int_E |f(t)|\,dt$ when $|E| > 0$. For each $f$ in $L_1(0, 1)$ define

$$f^E(t) = \begin{cases} f(t) & \text{if } t \text{ is in } E, \\ 0 & \text{otherwise.} \end{cases}$$

For $X \subset L_1(0, 1)$ let $X^E$ denote the closure in $L_1(E)$ of $\{f^E: f \text{ is in } x\}$.

A set $S \subset L_1(0, 1)$ is "complete on the set $E$" where $|E| > 0$ provided that $S^E = L_1(E)$. Otherwise, $S$ is "incomplete on the set $E$."

**Lemma 1.** *Let $X$ be a subspace of $L_1(0, 1)$ having a separable topological dual $X'$ and for which $X^E$ has finite codimension in $L_1(E)$, $|E| > 0$.*

*There then exists an $f$ in $X^E$ such that if $\{f_n\} \subset X$ and $\|f - f_n\|_E = o(1)$, then $\|f_n\| \longrightarrow \infty$. Such functions comprise all of $X^E$ except for a set of the first category in $X^E$.*

**Proof.** There can be no constant $M > 0$ such that given an $\epsilon > 0$ and an $f$ in $X^E$, there exists a $g$ in $X$ for which $\|f - g\|_E$ and $\|g\| \le M\|f\|_E$, where $M$ is independent of $\epsilon$ and $f$. For, suppose that such an $M$ exists. Let

$$R = \{f^E: f \in X, \|f\|_E \le 1, \text{ and } \|f\| \le M\}, \qquad S = \{f: f \in X^E, \text{ and } \|f\|_E \le 1\},$$

and let $T \in (X^E)'$. $T$ can be extended to an element of $L_1'(0, 1)$, and so there exists a bounded measurable function $h$ for which $T(f) = \int_E f(t)h(t)\,dt$.

Let $\|T\|$ denote the norm of $T$ as an operator on $X^E$, and let $N(T)$ denote the norm of $T$ as an operater on $X$. Since $R$ is by assumption dense in $S$,

$$MN(T) \ge \sup\{|T(f)|: f \in R\} = \sup\{|T(f)|: f \in S\} = \|T\|.$$

The nonseparability of $(X^E)'$ now implies the nonseparability of $X'$, a contradiction.

Now let $K_n = \{f \in X^E: \|f\|_E \le 1$ and there is a sequence $\{f_m\} \subset X$ for which $\|f_m - f\|_E = o(1)$ and $\|f_m\| \le n$, for all $m\}$.

Each $K_n$ is closed, convex, symmetric with respect to the origin in $X^E$. From what has been shown above, $K_n$ is not all of the unit ball in $X^E$ and is consequently nowhere dense. Thus $\bigcup_n K_n$ is of the first category, which is precisely the conclusion desired.

A set $S \subset L_1(0, 1)$ is "complete on a set $E$" where $|E| > 0$ provided that for any $f$ in $L_1(0, 1)$ and any $\epsilon > 0$, there is a $g$ in $S$ for which $\|f - g\|_E < \epsilon$.

**Theorem 2.** *Let $\{f_n\}$ be an orthonormal sequence of functions in $L_1(0, 1)$, $E \subset (0, 1)$, and $|E| > 0$, and let $X$ denote the closed linear span of $\{f_n\}$ in $L_1(0, 1)$.*

*If $X'$ is separable and $\{f_n^E\}$ is a Riesz system, then $\{f_n\}$ is incomplete on the set $E$.*

**Proof.** Suppose $\{f_n\}$ is complete on the set $E$. Let $f$ be as in the conclusion of Lemma 1 and let $\{g_n\} \subset X$ be such that $\|f - g_n\|_E = o(1)$. Lemma 1 implies that $\|g_n\| \to \infty$ but the hypothesized properties of $\{f_n\}$ imply that $\{\|g_n\|\}$ is bounded, a contradiction.

**Corollary 3.** *Every orthonormal system $\{f_n\}$ satisfying property* (B$'$) *contains a subsystem that is incomplete on every set of positive measure.*

**Proof.** Lemma 1.2.6$'$ of [G] assures the existence of a subsystem $\{f_{n_k}\}$ having the following property. Given $E \subset (0, 1)$, $|E| > 0$, there is a $k_0 = k_0(E)$ such that $\{f_{n_k}\}_{k>k_0}$ and $\{f_{n_k}^E\}_{k>k_0}$ are both Riesz systems. Theorem 2 now implies that $\{f_{n_k}\}$ is incomplete on the set $E$, proving the corollary.

As a special case, we consider the Walsh system [K, p. 132]. This system is easily seen to satisfy property (B$'$).

**Lemma 4.** *Let $\{w_n\}$ be a subsystem of the orthonormal system of Walsh possessing the property* $(*)$:

$(*)$ *if $b_1, \cdots, b_k, i_1, \cdots, i_k$ denote positive integers, then*

$$\int_0^1 w_{i_1}^{b_1}(t) \cdots w_{i_k}^{b_k}(t)\,dt = \begin{cases} 1, & \text{when all } b_i \text{ are even,} \\ 0, & \text{otherwise.} \end{cases}$$

*Let $1 \le p \le 2$, $E \subset [0, 1]$, $|E| > 0$, $0 < \epsilon < |E|/5$, and let $K \subset E$ with $|E \sim K| < \epsilon$. Then there are constants $A$, $B$, and $N$ depending only on $E$ such that for any $n > m > N$, and any sequence $a_m \cdots a_n$ of real numbers*

$$A\left(\sum_m^n a_i^2\right)^{1/2} \le \left(\int_K \left|\sum_m^n a_i w_i(t)\right|^p dt\right)^{1/p} \le B\left(\sum_m^n a_i^2\right)^{1/2}.$$

**Proof.** For any measurable set $S \subset [0, 1]$, define $S_{ij} \equiv \int_S w_i(t) w_j(t)\,dt$. Note that $\{w_i w_j\}_{i<j}$ is a subsystem of the Walsh system and that $S_{ij}$ is the $i$, $j$th Walsh-Fourier coefficient of the characteristic function of $S$. Bessel's inequality implies that $\Sigma_{i<j} S_{ij}^2 \le |S|$.

Since $\|\cdot\|_1 \le \|\cdot\|_p \le \|\cdot\|_2$, it is sufficient to establish the conclusion of the lemma for $p = 1$ and $p = 2$. For $p = 2$

$$\int_K \left(\sum_m^n a_i w_i(t)\right)^2 dt = \int_K \left(\sum_m^n a_i^2 + 2 \sum_{m \le i < j \le n} a_i a_j w_i(t) w_j(t)\right) dt$$

$$\le |K| \sum_m^n a_i^2 + 2\left(\sum_{m \le i < j \le n} a_i^2 a_j^2\right)^{1/2} \left(\sum_{m \le i < j \le n} K_{ij}^2\right)^{1/2}.$$

A direct calculation yields $(\sum_{m \le i < j \le n} a_i^2 a_j^2)^{1/2} = \sum_m^n a_i^2$. Also,

$$\left(\sum_{m \le i < j \le n.} K_{ij}^2\right)^{1/2} \le \left(\sum_{m \le i < j \le n} E_{ij}^2\right)^{1/2} + \left(\sum_{m \le i < j \le n} (E \sim K)_{ij}^2\right)^{1/2}$$

$$\le \left(\sum_{m \le i < j \le n} E_{ij}^2\right)^{1/2} + \epsilon.$$

Thus, if $N$ is chosen so that $\sum_N^\infty E_{ij}^2 < \epsilon$, and $n > m > N$, then

$$\int_K \left(\sum_m^n a_i w_i(t)\right)^2 dt \le |E| \sum_m^n a_i^2 + 2\left(\sum_m^n a_i^2\right)(2\epsilon)$$

$$= (|E| + 4\epsilon)\left(\sum_m^n a_i^2\right).$$

Similar considerations will establish the other half of the desired inequality. One obtains

$$\int_K \left(\sum_m^n a_i w_i(t)\right)^2 dt \ge (|E| - 5\epsilon)\left(\sum_m^n a_i^2\right),$$

and the case $p = 2$ is established.

For $p = 1$

$$\int_K \left|\sum_m^n a_i w_i(t)\right| dt \le |E|^{1/2} \left(\int_K \left(\sum_m^n a_i w_i(t)\right)^2 dt\right)^{1/2}$$

$$\le |E|^{1/2}(|E| + 4\epsilon)^{1/2}\left(\sum_m^n a_i^2\right)^{1/2}$$

provided that $n > m > N(\epsilon)$.

For the left-hand side we have:

$$\left(\sum_m^n a_k^2\right)^{1/2} \le (|E| - 5\epsilon)^{-1/2}\left(\int_K \left(\sum_m^n a_i w_i(t)\right)^2 dt\right)^{1/2}$$

$$= A\left(\int_K \left(\sum_m^n a_i w_i(t)\right)^{2/3}\left(\sum_m^n a_i w_i(t)\right)^{4/3} dt\right)^{1/2}.$$

An application of Hölder's inequality with $p = 3/2$ and $q = 3$ yields

$$(+) \qquad \left(\sum_m^n a_k^2\right)^{1/2} \le A\left(\int_K \left|\sum_m^n a_i w_i(t)\right| dt\right)^{1/3}\left(\int_K \left(\sum_m^n a_i w_i(t)\right)^4 dt\right)^{1/6}.$$

Observe that the Khintchine inequality [K, p. 130] is valid for any system of Walsh functions which possesses property (*). Taking this into account, set $p = 4$ in the Khintchine inequality. The last term in line (+) above can now be replaced by $B(\Sigma_m^n a_i)^{1/3}$, where $B > 0$ depends only on the choice of $p = 4$, without disturbing the sense of the inequality. The desired result is obtained when both sides of the resultant expression are cubed.

**Corollary 5.** *No system $\{w_k\}$ of Walsh functions possessing property (*) of Lemma 4 can be complete on any set of positive measure.*

**Proof.** Let $E \subset (0, 1)$, $|E| > 0$. Lemma 4 implies the existence of $N = N(E)$ for which $\{w_n\}_{n>N}$ and $\{w_n^E\}_{n>N}$ are both Riesz systems. Theorem 2 now implies that $\{w_n\}$ is incomplete on the set $E$.

A "quasi-basis" for a Banach space $(X, \|\cdot\|)$ is a double sequence $\{x_n, X_n\}$ of elements of $X$ and of continuous linear functionals, respectively, such that the series $\Sigma_1^\infty X_n(x)x_n$ converges in norm to $x$ for each $x$ in $X$. Such a structure may arise in the following way.

Suppose $\{y_n, Y_n\}$ is a Schauder basis for $L_p(0, 1)$, $1 \le p < \infty$. Let $|E| > 0$ and define for all $z$ in $L_p(0, 1)$

$$z^E(t) = \begin{cases} z(t), & t \text{ in } E, \\ 0, & \text{otherwise.} \end{cases}$$

For each given $\phi$ in $L_p'(0, 1)$, define $\phi^E(z) = \phi(z^E)$ for any $z$ in $L_p(0, 1)$.

It is easily verified that $\{y_n^E, Y_n^E\}$ is a quasi-basis for $L_p(E)$. This double sequence shall be called the "restriction of $\{y_n, Y_n\}$ to the set $E$."

The quasi-basis $\{x_n, X_n\}$ is "unconditional" if $\Sigma_1^\infty X_n(x)x_n$ converges unconditionally for every $x$ in $X$. Otherwise, the quasi-basis is "conditional." The restriction of an unconditional basis for $L_p(0, 1)$ to any set $E$, $|E| > 0$, is clearly an unconditional quasi-basis for $L_p(E)$. A problem that arises in this context is the determination of the sets $E$, if there are any, for which a given conditional basis for $L_p(0, 1)$ restricts to an unconditional quasi-basis for

$L_p(E)$. Two special cases are considered below: it turns out that no such sets exist for the Walsh system [K, p. 132] in $L_p(0, 1)$, $1 < p < \infty$, or for the Haar system [K, p. 120] in $L_1(0, 1)$.

Let $(X, \|\cdot\|)$ denote a Banach space of equivalence classes of Lebesgue measurable functions on [0, 1] (see [L, p. 66] and [Z, Chapter 15]). Such spaces may be defined by specifying a certain class $H$ of elements of $L_1(0, 1)$ and defining

$$\|x\| = \sup\left\{\int_0^1 |x(t)h(t)|\,dt \colon h \in H\right\} \quad \text{and} \quad X = \{x\colon \|x\| < \infty\}.$$

Various conditions may be imposed on $H$ in order that $(X, \|\cdot\|)$ turns out as a Banach space.

**Theorem 6.** *Given a quasi-basis $\{x_n, X_n\}$ for $(X, \|\cdot\|)$. Let $\{r_n\}$ denote the Rademacher system and define*

$$C = \left\{\theta\colon \sum_1^\infty r_k(\theta)X_k(x)x_k \text{ converges for all } x \text{ in } X\right\}.$$

*Let $Z_N = \{x \in X\colon X_1(x) = \cdots = X_N(x) = 0\}$ and let $G(x) = \|(\Sigma_1^\infty X_n^2(x)x_n^2)^{1/2}\|$.*

*Then $C$ is a Borel set, and if $|C| > 0$ there is an $N$ for which the identity mapping $(Z_N, \|\cdot\|) \to (Z_N, G(\cdot))$ is continuous.*

**Proof.** Let $x(n, m, \theta, t) = \Sigma_n^m r_k(\theta)X_k(x)x_k(t)$. For each $(n, m)$, $\|x(n, m, \theta, \cdot)\|$ is a step function of $\theta$ and

$$C = \left\{\theta\colon \operatorname*{Lim\,sup}_{n, m\to\infty} \|x(n, m, \theta, \cdot)\| = 0\right\}.$$

$C$ is therefore a Borel set.

For each $\theta$ in $C$, the sequence of linear operaters $T_{n\theta}(\cdot)$ defined by

$$T_{n\theta}(x) = \sum_1^n r_k(\theta)X_k(x)x_k$$

is uniformly bounded by virtue of the Banach-Steinhaus theorem. The set $S_M \equiv \{\theta\colon \|T_{n\theta}\| \le M \text{ for all } n\}$ is clearly a Borel set, and $\bigcup(S_M \cap C) = C$. Since $|C| > 0$, there is a constant $K > 0$ for which $|S_K \cap C| > 0$. Define $S \equiv S_K \cap C$, and for each $\theta$ in $S$ let $T_\theta(x) \equiv \lim_n T_{n\theta}(x)$. Then,

$$(1) \qquad \|T_\theta(x)\| = \lim_n \|T_{n\theta}(x)\| \le M\|x\| \quad \text{for all } \theta \in S,$$

and $S$ is seen to be a Borel set of positive measure

For any $\epsilon > 0$ there is, by definition of $\|\cdot\|$, an $h$ in $H$ such that

$$G(x) \leq \int_0^1 \left(\sum_1^\infty X_k^2(x) x_k^2(t)\right)^{1/2} |h(t)|\, dt + \epsilon. \tag{2}$$

By Lemma 4, if $F \subset S$ is of sufficiently large measure, there is a $B > 0$ and an integer $N > 0$, depending only on $S$, such that

$$B\left(\sum_N^n X_k^2(x) x_k^2(t)\right)^{1/2} |h(t)| \leq \int_F |x(N, n, \theta, t) h(t)|\, d\theta \tag{3}$$

for every $x$ in $X$, and for every $n > N$.

For each $\theta$ in $S$,

$$\operatorname*{Lim}_n \int_0^1 |x(N, n, \theta, t) h(t)|\, dt = \int_0^1 |x(n, \infty, \theta, t) h(t)|\, dt. \tag{4}$$

By Egoroff's theorem there is $F \subset S$ in which the convergence in (4) is uniform, and having measure large enough so that line (3) obtains. We may integrate (4) with respect to $\theta$ over the set $F$, and the order of integration on the left side may be reversed. This yields

$$\operatorname*{Lim}_n \int_0^1 \left[\int_F |x(N, n, \theta, t) h(t)|\, d\theta\right] dt = \int_F \left[\int_0^1 |x(N, \infty, \theta, t) h(t)|\, dt\right] d\theta. \tag{5}$$

Assume now that $x$ is in $Z_N$, and combine (1), (2), (3), and (5):

$$G(x) \leq \frac{1}{B} \int_F \left[\int_0^1 |x(N, \infty, \theta, t) h(t)|\, dt\right] d\theta + \epsilon \leq \frac{1}{B} \int_F \|T_\theta(x)\|\, d\theta + \epsilon \leq \frac{M}{B} \|x\| + \epsilon$$

for all $x$ in $Z_N$, and for any $\epsilon > 0$.

The norm $G(\cdot)$ and unconditional convergence have been discussed previously in [O], and in the case that $\{x_n, X_n\}$ is an unconditional basis it has been shown in [SZ] that the norms $G(\cdot)$ and $\|\cdot\|$ are equivalent.

Theorem 6 can be used to partially extend Lemma 4 to all of the spaces $L_p(0, 1)$, $1 < p < \infty$:

**Corollary 7.** *Let $\{w_n\}$ be a subsystem of the Walsh system possessing the property* (*) *described in Lemma* 4, *and let* $1 \leq p < \infty$, $|E| > 0$. *Then there are constants $A$, $B$, and $N$ depending only on $E$ such that for any $n > m > N$ and any sequence $a_m \cdots a_n$ of real numbers*

$$A\left(\sum_m^n a_i^2\right)^{1/2} \leq \left(\int_E \left|\sum_m^n a_i w_i(t)\right|^p dt\right)^{1/p} \leq B\left(\sum_m^n a_i^2\right)^{1/2}.$$

**Proof.** For $2 \leq p < \infty$, let $N$ be determined by the set $E$ as in Lemma 4, and let $Y$ be the closed linear span of $\{w_i^E\}_{i \geq N}$ in $L_p(E)$. If continuous linear functionals $W_i$ on $Y$ can be found for which $\{w_i^E, W_i\}_{i \geq N}$ is an unconditional basis for $Y$, the desired conclusion will then follow from Lemma 4, Theorem 6 and the fact that $\|\cdot\|_p \geq \|\cdot\|_2$.

Lemma 4 implies that for each $x$ in $Y$, there is a unique sequence $\{a_i\}$ in $l_2$ for which the series $\Sigma_{i \geq N} a_i w_i^E$ converges unconditionally to $x$ in the norm of $L_2(0, 1)$. The Khintchine inequality with the functions $w_i$ substituted for Rademacher's functions [ see proof of Lemma 4 above] implies that $\Sigma_{i \geq N} a_i w_i \equiv \bar{x}$ converges unconditionally in the norm of $L_p(0, 1)$ and that the mapping $x \longrightarrow \bar{x}$ taking $Y \longrightarrow L_p(0, 1)$ is continuous in that norm.

Clearly, the functionals $W_i$ defined by $W_i(x) \equiv \int_0^1 w_i(t)\bar{x}(t)\,dt$ are as desired.

The case $1 \leq p \leq 2$ is a special case of Lemma 1, proving the corollary.

It is known that the Walsh system is a conditional basis for $L_p(0, 1)$, $1 < p < \infty$, $p \neq 2$, [O] and [P]. Also, the Haar system is a conditional basis for $L_1(0, 1)$, [D] and [M]. The conditionality of these systems is preserved under restriction to a set of positive measure.

**Theorem 8.** *Let $\{w_n, W_n\}$ denote the quasi-basis for $L_p(E)$, $1 < p < \infty$, obtained by restricting the Walsh system to a set $E$, $|E| > 0$.*

*If $p \neq 2$, then $\{w_n, W_n\}$ is a conditional quasi-basis for $L_p(E)$. If $p = 2$, then this quasi-basis is unconditional.*

**Proof.** If $p = 2$, the Walsh system is an unconditional basis for $L_p(0, 1)$, and so $\{w_n, W_n\}$ is an unconditional quasi-basis for $L_p(E)$.

The remaining cases $1 < p < 2$ and $2 < p < \infty$ are considered separately. For $p < 2$, let $x \in L_p(E) \sim L_2(E)$. Then

$$G(x) = \left\| \left( \sum_1^\infty W_n^2(x) w_n^2 \right)^{1/2} \right\|_p = \left( \sum_1^\infty W_n^2(x) \right)^{1/2} |E| = \infty.$$

It follows that the series $\Sigma_1^\infty W_n(x) w_n$ is conditionally convergent [O].

For $p > 2$, let $y \in L_q(E)$ where $1/p + 1/q = 1$. Were $\{w_n, W_n\}$ an unconditional quasi-basis for $L_p(E)$, then

$$\epsilon(x) \equiv \operatorname*{Lim}_n \sum_1^n W_k(x) \epsilon_k w_k$$

would exist for every $x$ in $L_p(E)$ and for every sequence $\epsilon \equiv \{\epsilon_k\}$ where $\epsilon_k = \pm 1$. Then

$$(y, \epsilon(x)) = \sum_1^\infty W_k(x)\epsilon_k(w_k, y) = \sum_1^\infty W_k(y)\epsilon_k(w_k, x)$$

would converge for all $x$ and $\epsilon$.

The series $\Sigma_1^\infty W_k(y)w_k$ would then be subseries convergent in the weak topology on $L_q(E)$ and hence subseries convergent in the norm topology [D, p. 60]. $\{w_k, W_k\}$ would then be an unconditional quasi-basis for $L_q(E)$, which is impossible since $1 < q < 2$.

**Theorem 9.** *Let $\{h_{np}^E, H_{np}^E\}$ denote the quasi-basis for $L_1(E)$ obtained by restricting the Haar system to a set $E$, $|E| > 0$.*

*Then this system is a conditional quasi-basis for $L_1(E)$.*

**Proof.** Theorem 6 implies that if $\{h_{np}^E, H_{np}^E\}$ were an unconditional quasi-basis for $L_1(E)$, then the norm $\|\cdot\|$ would dominate the norm $G(\cdot)$ on some space $Z_N = \{x \in L_1(E): H_{np}(x) = 0 \text{ for } n \leq N,\ p = 0, 1, \cdots, 2^n - 1\}$. But this cannot be the case: a sequence $\{y_p\}$ is constructed for which

(A) $\{y_p\} \subset Z_N$,

(B) $\{y_p\}$ is bounded in $(L_1(E), \|\cdot\|)$,

(C) $\{y_p\}$ is unbounded in $(L_1(E), G(\cdot))$.

Let $p$ be a fixed positive integer, $p > 4$, and let $\epsilon < 1/p$. Since almost every point of $E$ is a point of metric density 1, there exists a sequence $\{I_k\}_{k=0}^p$ of dyadic intervals, each of which is the support of a Haar function which shall be denoted by $h_k$, and for which

$I_{k+1}$ is the left half of $I_k$,

$|I_k| = 2^{-n-k}$, $k = 0, 1, \cdots, p$,

$0 \leq 1 - |E \cap I_k|/|I_k| < \epsilon$, $k = 0, \cdots, p$.

Let $E_k = E \cap I_k$ and split $E_k$ into a left half $L_k$ and a right half $R_k$ of equal measure. Define, for $k = 0, 1, \cdots, p$,

$$Y_k(t) = \begin{cases} \sqrt{2^{n+k}} & \text{if } t \text{ is in } L_k, \\ -\sqrt{2^{n+k}} & \text{if } t \text{ is in } R_k, \\ 0 & \text{otherwise,} \end{cases}$$

and let $y_p = \Sigma_{k=0}^p \sqrt{2^{n+k}}Y_k$.

The proof shall be completed when it is demonstrated that the sequence $\{y_p\}$ possesses properties (A), (B) and (C) above. This necessitates estimates on $H_i^E(y_p) = \Sigma_{k=0}^p \sqrt{2^{n+k}}H_i(Y_k)$.

The function $Y_k$ resembles the Haar function $h_k$ restricted to the set $E$. In particular, if $e_k$ is defined by the formula

$$Y_k(t) = h_k^E(t) + e_k(t)$$

then

$$|\text{supp } e_k| \leq \tfrac{1}{2}(|I_k| - |E_k|) \leq \epsilon 2^{-n-k-1},$$
$$|e_k(t)| \leq 2\sqrt{2^{n+k}}.$$

Estimation of $H_i^E(Y_k)$ for $k \geq i$:

$$|H_i^E(b_i)| = 2^{n+i}|E_i|$$

and

$$|H_i^E(e_i)| \leq \sqrt{2^{n+i}} \max |e_i(t)| \, |\text{supp } e_i| = 2^{n+i+1}|\text{supp } e_i| \leq \epsilon.$$

Thus,

$$|H_i^E(Y_i)| \geq 2^{n+i}|E_i| - \epsilon \geq 1 - 3\epsilon.$$

If $k > i$, then supp $Y_k$ is contained in that part of $E_i$ on which $b_i$ is of constant sign. In this case, $H_i^E(Y_k) = 0$.

Estimation of $H_i^E(Y_k)$ for $k < i$:

$$|H_i^E(b_k)| \leq \sqrt{2^{n+i}}\sqrt{2^{n+k}}(|I_i| - |E_i|) \leq \epsilon\sqrt{2^{n+i}}\sqrt{2^{n+k}}2^{-n-i} = \epsilon\sqrt{2^{k-i}},$$
$$|H_i^E(e_k)| \leq 2\sqrt{2^{n+i}}\sqrt{2^{n+k}}|\text{supp } e_k \cap E_i|.$$

If $k < i-1$, supp $e_k \cap E_i = \varnothing$ and then $H_i^E(e_k) = 0$. Otherwise, $|H_i^E(e_{i-1})| \leq 2\sqrt{2^{n+i}}\sqrt{2^{n+i-1}}2^{-n-i}\epsilon = \epsilon\sqrt{2}$. This gives

$$|H_i^E(Y_k)| \leq \epsilon\sqrt{2^{k-i}} \quad \text{if } k < i-1,$$
$$|H_i^E(Y_{i-1})| \leq \epsilon\sqrt{2} + \epsilon/\sqrt{2} \leq 3\epsilon.$$

Upon combining the above estimates in $H_i^E(Y_k)$,

$$\begin{aligned}
|H_i^E(y_p)| &= \left|\sum_{k=0}^{p} \sqrt{2^{n+k}} H_i^E(Y_k)\right| \\
&\geq |\sqrt{2^{n+i}} H_i^E(Y_i)| - \sqrt{2^{n+i-1}}|H_i^E(Y_{i-1})| \quad \sum_{k=0}^{i-2} \sqrt{2^{n+k}}|H_i^E(Y_k)| \\
&\geq \sqrt{2^{n+i}}(1-3\epsilon) - 3\epsilon\sqrt{2^{n+i-1}} - \epsilon \sum_{k=0}^{i-2} \sqrt{2^{n+k}}\sqrt{2^{k-i}} \\
&\geq \sqrt{2^{n+i}}(1-6\epsilon) - \epsilon\sqrt{2^{n-i}} \sum_{k=0}^{i-2} 2^k \\
&\geq \sqrt{2^{n+i}}(1-6\epsilon) - \epsilon\sqrt{2^{n+i}} = \sqrt{2^{n+i}}(1-7\epsilon).
\end{aligned}$$

Let $\|\cdot\|_E$ denote the $L_1$-norm of functions restricted to $E$. An application of the estimate above then yields

$$G(y_p) \geq \left\| \left( \sum_{i=0}^{p} [H_i^E(Y_p)]^2 b_i^2 \right)^{1/2} \right\|_E \geq A \left\| \left( \sum_{i=0}^{p} 2^{n+i} b_i^2 \right)^{1/2} \right\|_E$$

$$\geq B \sum_{i=0}^{p} 2^{-n-i} \left( \sum_{k=0}^{i} 2^{2(n+k)} \right)^{1/2} \geq Cp.$$

Statement (C) is thereby established. Moreover,

$$\|y_p\| \leq \left\| \sum_{k=0}^{p} \sqrt{2^{n+k}} b_k \right\| + \left\| \sum_{k=0}^{p} \sqrt{2^{n+k}} e_k \right\| \leq 2 + \sum_{k=0}^{p} \sqrt{2^{n+k}} \|e_k\| \leq 2 + p\epsilon \leq 3.$$

This establishes statement (B). Given any $N$, it is clear that the intervals $I_k$ can be so chosen that the functions $y_k$ are all elements of $Z_N$. Then $y_p$ will also be an element of $Z_N$, and (A) is established.

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